Lesson

3-4 Symmetries of Graphs

Vocabulary

reflection-symmetric
axis of symmetry
line of symmetry
symmetric about a point
point symmetry
center of symmetry
even function
odd function

BIG IDEA Rules for functions can be used to determine symmetries of graphs and vice versa.

Mental Math

What is the least number of symmetry lines the figure can have?

a. parabola

b. rectangle

c. equilateral triangle

d. circle

Recall from your previous courses that a figure is **reflection-symmetric** if and only if the figure can be mapped onto itself by a reflection over some line l. The reflecting line l is called the **axis** or **line of symmetry** of the figure. A line of symmetry can be any line in the plane. Similarly, a figure is **symmetric about point** p or has **point symmetry** if and only if the figure can be mapped onto itself under a rotation of 180° around P. The point P is called a **center of symmetry**.

Because graphs are sets of points, these definitions apply to graphs. Any symmetry of a figure implies that one part of the figure is congruent to another part. So when a graph has symmetry, the symmetry shortens the time in drawing the graph and helps in studying it.

Activity 1

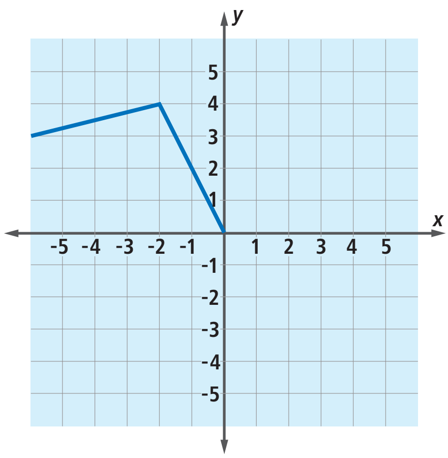
The diagram at the right shows half of a graph.

Step 1 Copy the diagram. Draw the other half of the graph so that the result is point-symmetric about the origin. Label this half A.

Step 2 Draw the other half of the original graph so that the result is symmetric with respect to the y-axis. Label this half B.

Step 3 Draw the other half of the original graph so that it is symmetric over the x-axis. Label the graph C.

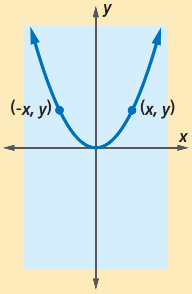
Step 4 What symmetries does the union of graphs A, B, and C and the original graph possess?



Recall also that the reflection image of (x, y) over the x-axis is $(x, -y)$, and the reflection image of (x, y) over the y-axis is $(-x, y)$. Also the image of (x, y) under a rotation of 180° about the origin is $(-x, -y)$. Combining these facts with the definitions of reflection, symmetry, and point symmetry yields three theorems.

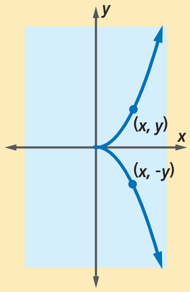
Theorem (Symmetry over *y*-axis)

A graph is symmetric with respect to the *y*-axis if and only if for every point (x, y) on the graph, $(-x, y)$ is also on the graph.



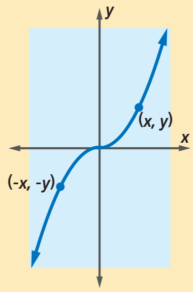
Theorem (Symmetry over *x*-axis)

A graph is symmetric with respect to the *x*-axis if and only if for every point (x, y) on the graph, $(x, -y)$ is also on the graph.



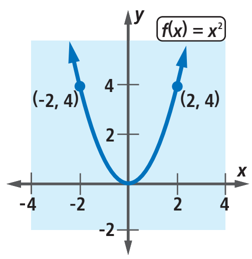
Theorem (Symmetry about the Origin)

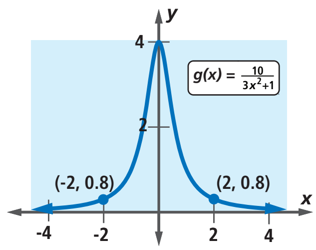
A graph is symmetric to the origin if and only if for every point (x, y) on the graph, $(-x, -y)$ is also on the graph.



Proving That a Graph Has Symmetry

One point can be a counterexample that shows a graph does not have a certain type of symmetry. But one point cannot determine that a general relationship holds. The graphs of $f(x) = x^2$ and $g(x) = \dfrac{10}{3x^2 + 1}$ seem symmetric with respect to the *y*-axis. But this is not a proof.





To show that the graph of an equation is symmetric, you use algebra and test the general point (x, y).

Example 1

Prove that the graph of $f(x) = \dfrac{1}{x^2 + 1}$ is symmetric with respect to the *y*-axis.

Solution To show that the graph of *f* is symmetric with respect to the *y*-axis, you need to show that for all (x, y) on the graph, $(-x, y)$ is also on the graph. In other words, you need to prove $f(x) = f(-x)$ for all *x* in the domain of *f*.

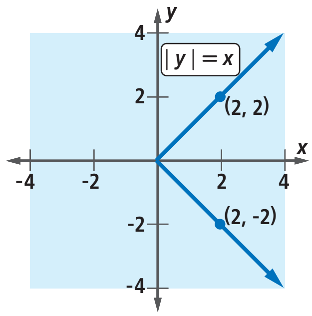
$$f(x) = \frac{10}{3x^2 + 1}$$

$$f(-x) = \frac{10}{3(-x)^2 + 1} = \frac{10}{3x^2 + 1}$$

So $f(x) = f(-x)$. Therefore the graph of $f(x) = \dfrac{10}{3x^2 + 1}$ is symmetric with respect to the y-axis.

Using Symmetry to Aid in Graphing

The ideas of symmetry apply to graphs of relations that are not functions. For instance, the graph of $x = |y|$ is symmetric over the x-axis, since if $x = |y|$, then $x = |-y|$.

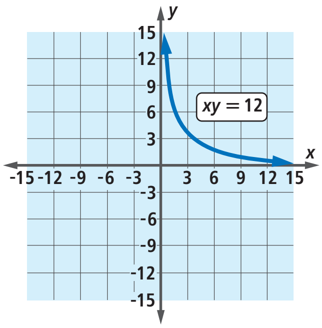


Activity 2

The part of the graph of $xy = 12$ that is in Quadrant I is shown at the right.

Step 1 Test to see if the equation is symmetric with respect to the y-axis, the x-axis, or the origin.

Step 2 Use the results to complete the graph.



Even and Odd Functions

In previous courses and Lesson 2-7, you studied power functions with equations such as $y = x^2$, $y = x^3$, or $y = x^4$. The power functions f with $f(x) = ax^n$, where $a \neq 0$ and n is even, can all be proved to be symmetric over the y-axis. For this reason, any function whose graph is symmetric with respect to the y-axis is called an *even function*.

Definition of Even Function

A function is an **even function** if and only if for all values of x in its domain, $f(-x) = f(x)$.

By the method of Example 1, the graphs of the power functions f with $f(x) = ax^n$, where $a \neq 0$ and n is odd, can all be proved to be symmetric about the origin. For this reason, a function whose graph is symmetric about the origin is called an *odd function*.

Definition of Odd Function

A function f is an **odd function** if and only if for all values of x in its domain, $f(-x) = -f(x)$.

STOP QY

If you are not sure if a function is even, odd, or neither, then a graphing utility or CAS may help you decide.

QY

Look at the graphs for the theorems on page 173 of this lesson.

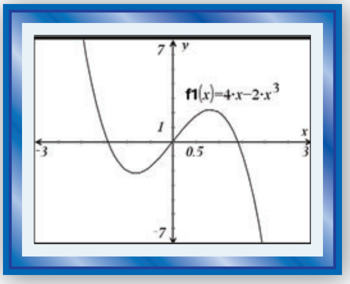
a. Which graph shows an even function?

b. Which graph shows an odd function?

Example 2

Determine whether the function $f: x \longrightarrow 4x - 2x^3$ is odd, even, or neither. If it appears to be even or odd, prove it.

Solution 1 Draw a graph to see if f appears to be even or odd. A graph of f is shown at the right.

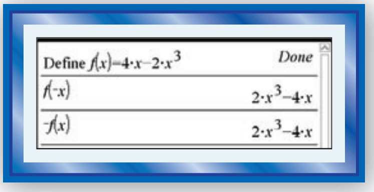


The graph appears to be symmetric about the origin, so f seems to be an odd function. To prove this, suppose (x, y) is on the graph. That is, $y = f(x) = 4x - 2x^3$. Now consider $f(-x)$.

$$\begin{aligned} f(-x) &= 4(-x) - 2(-x)^3 \\ &= -4x + 2x^3 \\ &= -(4x - 2x^3) \end{aligned}$$

Thus, $f(-x) = -f(x)$ for all x, so f is an odd function.

Solution 2 Define $f(x) = 4x - 2x^3$ on a CAS. Evaluate $f(-x)$ and $-f(x)$, as shown at the right. The expression for $f(-x)$ is the same as for $-f(x)$. For all x, $f(-x) = -f(x)$, so f is odd.



Using the Graph-Translation Theorem to Find Symmetries

You have seen graphs with symmetry related to the x-axis, the y-axis, and the origin. However, a line or point of symmetry may be located in other positions. If the graph is the translation image of a familiar graph, the symmetry of the known graph can give information about symmetry in the image.

Example 3

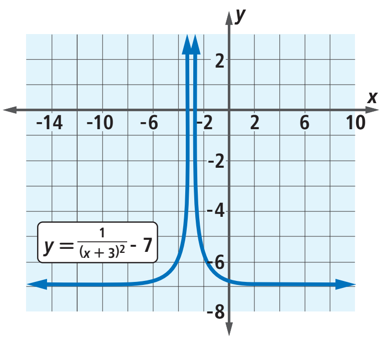
Consider the function F with $y = F(x) = \frac{1}{(x+3)^2} - 7$.

a. Give equations for the asymptotes of its graph.

b. Describe any lines or points of symmetry.

Solution

a. Rewrite the equation as $y + 7 = \frac{1}{(x+3)^2}$. This shows that, by the Graph-Translation Theorem, the graph of F is the image of the graph of $y = \frac{1}{x^2}$ under the translation $T(x, y) = (x - 3, y - 7)$. The graph of the parent function has asymptotes $x = 0$ and $y = 0$. Each asymptote is translated 3 units to the left and 7 units down. So, the asymptotes of F are $x = -3$ and $y = -7$.



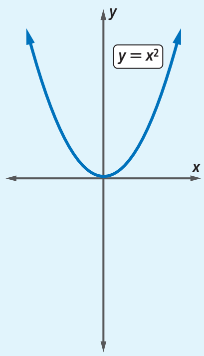
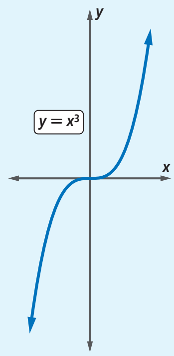
b. Since the graph of $y = \frac{1}{x^2}$ is symmetric over the y-axis, the graph of F is symmetric over the vertical line $x = -3$. Indeed, this line is the translation image of $x = 0$ under $T: (x, y) \longrightarrow (x - 3, y - 7)$.

Check Sketch a graph of F.

Properties of Graphs of Translation Images

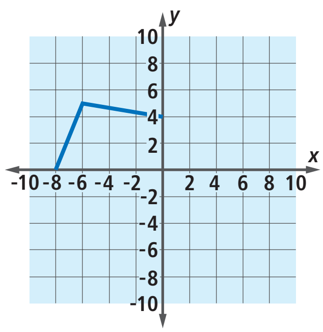
In general, if *f* is a function and each point (x, y) on its graph is mapped to $(x + h, y + k)$, then the graph of the image is congruent to the graph of the preimage, and all key points and lines are also mapped under this translation. Specifically, lines of symmetry map to lines of symmetry, maxima to maxima, minima to minima, vertices to vertices, symmetry points to symmetry points, and asymptotes to asymptotes.

This chart summarizes the characteristics of even and odd functions.

	Even Function	Odd Function
Symmetry	symmetric over y-axis	symmetric about the origin
Transformation	$(x, y) \longrightarrow (-x, y)$	$(x, y) \longrightarrow (-x, -y)$
Function Notation	$f(-x) = f(x)$	$f(-x) = -f(x)$
Sample Graph and Equation		

Questions

COVERING THE IDEAS

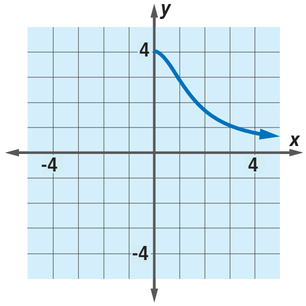


1. The part of a graph that is in Quadrant II is shown. The graph is symmetric over the x-axis and passes through only two quadrants.
   a. Copy the graph and complete it.
   b. The point $(-6, 5)$ is on the graph. Use symmetry to find another point on the graph.

In 2–4, suppose z is a function that includes $(4, -2)$. What other point must be included in the relation if z has the stated property?

2. z is odd.
3. z is even.
4. The graph of z is symmetric with respect to the x-axis.
5. The point $(-3, -9)$ satisfies the equation $y = x \cdot |x|$. Use this point to test whether the graph of $y = x \cdot |x|$ appears to be symmetric
   a. over the y-axis. **b.** over the x-axis. **c.** about the origin.
6. Prove that the function f defined by $f: x \to 6x^{-1}$ is an odd function.

7. The part of the graph of $y = \frac{8}{2 + x^2}$ that lies in Quadrant I is at the right. Test for symmetry with respect to the y-axis, x-axis, and origin. Then, complete the graph.



8. For each type of function in the left column, name two properties from the right column which match.

a.	even function	i.	The graph is symmetric about the origin.
b.	odd function	ii.	If (x, y) is in the function, so is $(-x, y)$.
		iii.	If (x, y) is in the function, so is $(-x, -y)$.
		iv.	The graph is symmetric over the y-axis.
		v.	The graph is symmetric over the x-axis.
		vi.	If (x, y) is in the function, so is $(x, -y)$.

In 9 and 10, an equation of a function is given. Tell if the function is odd, even, or neither.

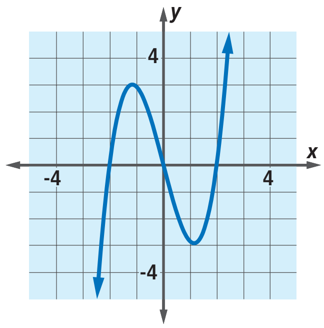
9. $f(x) = -246x$

10. $g(x) = \frac{1}{2}x^3 + 3$

11. Consider the function $f: x \rightarrow \frac{1}{x - 6} + 1$.
   a. Sketch a graph of $y = f(x)$.
   b. Give equations for the asymptotes of the graph.
   c. How are these asymptotes related to the asymptotes of the parent function?

APPLYING THE MATHEMATICS

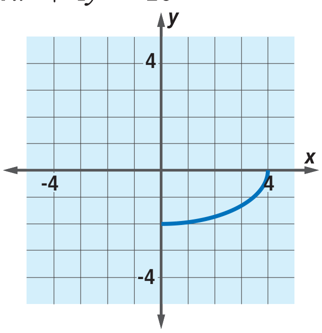
12. At the right is the graph of a function f that is point-symmetric with respect to the origin. Sketch the graph that is the translation image of the graph of f with $(-6, 4)$ for a center of symmetry.



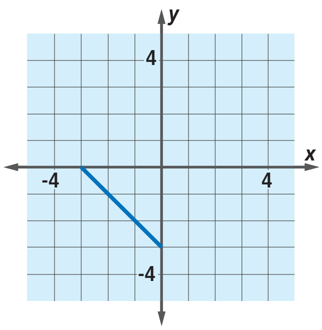
13. Give a counterexample to prove that the function f with $f(t) = t^3 - 2$ is not an even function.

In 14 and 15, one quadrant of the graph of the given equation is shown. Use x- and y-axis symmetry to complete the graph.

14. $x^2 + 4y^2 = 16$



15. $|x| + |y| = 3$



16. Recall from geometry that a circle whose center is at the origin has an equation of the form $x^2 + y^2 = r^2$. Prove algebraically that the circle with equation $x^2 + y^2 = 9$ has all three types of symmetry discussed in this lesson.

17. Prove that if f and g are odd functions, so is $f + g$. *Hint*: Consider $(f + g)(x)$ and $(f + g)(-x)$.

18. Use the function g with $g(x) = x^3$ as a parent function. The functions h and j with $h(x) = -x^3$ and $j(x) = x^3 + 2$ are related to it.
   a. Sketch the graphs of all three functions.
   b. Determine if each function is odd, even, or neither.
   c. Describe a transformation that maps g onto h.
   d. Describe a transformation that maps g onto j.
   e. Describe the symmetries of the three graphs. If there is reflection symmetry, give the equation of the line of symmetry. If there is point symmetry, give the coordinates of the center of symmetry.

REVIEW

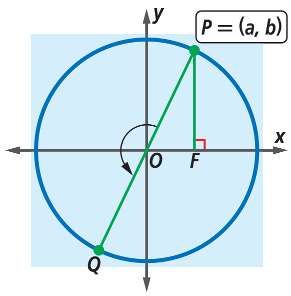
19. The class results of a test on *Moby Dick* in American Literature were $\bar{x} = 67$ and $s = 13$. The teacher decides to add 4 points to each score. **(Lesson 3-3)**
   a. What is the class mean after this transformation?
   b. What is the class standard deviation after this transformation?

A whale of a tale
Moby Dick was a sperm whale.

20. ***Skill Sequence*** If $t_1 = 10$, $t_2 = -1$, $t_3 = 16$, and $t_4 = 6$, evaluate the expression. **(Lessons 3-3, 1-2)**
   a. $\sum_{i=1}^{4} t_i - 2$ b. $\sum_{i=1}^{4} (t_i - 2)$ c. $-2\sum_{i=1}^{4} t_i$

21. Suppose the translation T: $(x, y) \to (x + 2, y - 5)$ is applied to the graph of $y = x^5$. **(Lesson 3-2)**
   a. Find an equation for the image.
   b. Are the graphs of the preimage and image congruent? Explain why or why not.

In 22–25, use the circle at the right. $P = (a, b)$ and Q is the image of P under a 180° rotation around O. Express each of the following in terms of a and b. (Previous Course)



22. OF
23. OP
24. x-coordinate of Q
25. y-coordinate of Q

EXPLORATION

26. a. Can the graph of a function be mapped onto itself by a rotation whose magnitude is not 180° or 360°? If so, give some examples. If not, explain why not.
   b. Consider the question of Part a for graphs of relations which are not functions.

QY ANSWER

a. the graph with symmetry over the y-axis

b. the graph with symmetry about the origin